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## ABSTRACT

Estimation problems, and filtering among them, are basically concerned with extracting the best information from inaccurate observation of signals. P~~h~~erhaps the earliest roots of this type of problems go back to the least squares estimation at the time of Galileo Galilei in 1632 and Gauss in 1795. The relatively modern and more general development of least-squares estimation in stochastic processes is marked by the work of A.N. Kolmogorov and N. Wiener in the 1940's. Most recently, and due to vast research and development of the space age, the estimation theory experienced a new outlook. This was marked by the work of P. Swerling in 1958 and 1959 in connection with satellite tracking, and the work of R. Kalman using state space approach. Kalman's work ~~[1]~~ had the impact of greatly popularizing and spreading the estimation theory in different fields of applications. Also, works by Stratonovich and Kushner are among the recent developments of the subject.

# FILTERING OF SYSTEMS WITH NONLINEARITIES

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## I. Introduction

Estimation problems, and filtering among them, are basically concerned with extracting the best information from inaccurate observation of signals. Perhaps the earliest roots of this type of problems go back to the least squares estimation at the time of Galileo Galilei in 1632 and Gauss in 1795. The relatively modern and more general development of least-squares estimation in stochastic processes is marked by the work of A.N. Kolmogorov and N. Wiener in the 1940's. Most recently, and due to vast research and development of the space age, the estimation theory experienced a new outlook. This was marked by the work of P. Swerling in 1958 and 1959 in connection with satellite tracking, and the work of R. Kalman using state space approach. Kalman's work [1] had the impact of greatly popularizing and spreading the estimation theory in different fields of applications. Also, works by Stratonovich and Kushner are among the recent developments of the subject.

From the control theory point of view, the problem of estimating the state dynamical systems plays an important role. Very often the optimal control law sought for a dynamical system is some sort of a feedback of its state. Take for example the control of a chemical process, a nuclear reactor, maneuvering of a space craft, guidance and navigation problems, and the problem of control and suppression of structural vibrations. Also, sometimes, it is of interest to know the state of a dynamic system. Take for example the tracking of moving objects like satellites in orbits, and enemy missiles. These are just a few examples of the application of this knowledge.

Fundamentally, the conditional probability density of the state conditioned on available observations holds the key for all kinds of state estimators. The case of the linear dynamical system, with measurements linear in the state variables, in the presence of additive Gaussian noise, and under the assumption of full knowledge of the system parameters and noise statistics, has been optimally solved. In that particular case, the conditional probability density is Gaussian. A Gaussian density is characterized by only two quantities, namely, its mean and covariance. Therefore, the optimal linear filter has a finite state, the conditional mean and the conditional covariance, and is widely known as the Kalman or the Kalman-Bucy filter [1], [2], [3], and [4]. The Kalman filter provides the minimum variance unbiased estimates. Also, the filter structures is linear, its gain and covariance can be processed independently of the estimate even before receiving the observations. These features make the Kalman filter desirable and easy to implement.

Unlike the linear case, the situation for nonlinear systems is completely different. The conditional probability density is no longer Gaussian even though the acting noise is itself Gaussian. In this case the evolution of the conditional probability density is governed by a stochastic integral-partial differential equation, Kushner's equation, or equivalently by an infinite set of stochastic differential equations for the moments of the density function [3], [42], and [43]. Therefore, the truly optimal nonlinear filter is of infinite dimensionality, and consequently is of no practical interest. Therefore, practical suboptimal finite dimensional filters are very much needed.

Inspired by Kalman's results, a great deal of research effort has been directed towards extending the linear results and developing practical schemes for nonlinear filters. Developments have relied on two main approaches.

The first approach is based on the linearization of system nonlinearities around a nominal trajectory using Taylor's series expansion. Performing the expansion up to the first order terms results in the linearized filter [3], and [11]. The approach can further be improved by linearizing, again up to a first order, about the most recent estimate. Relinearization is performed as more recent estimates become available. By so doing the well known extended Kalman filter (EKF), [3], is obtained. The Taylor's series expansion can be carried up to the second order terms. In this case, with some assumptions on the conditional probability density function, second order filters are obtained. Among these are the truncated second order filter, the Gaussian second order filter, and the modified second order filter (M2-F). These second order filters are presented in [3], and [11].

In the second approach the conditional probability density function is approximated using several techniques. The Gaussian sum approximation is used in [33], and [34]. In this case the conditional probability density is approximated by a finite weighted sum of Gaussian densities with different means and covariances. Since the Kalman filter is a Gaussian density synthesizer, then the resulting Gaussian sum filter is actually a bank of Kalman filters working in parallel. Each one is properly tuned in terms of system parameters and its output is properly weighted and summed to other filters' outputs to produce the state estimate. The approach has been used extensively by many authors to treat the estimation problem of linear systems with unknown parameters e.g. [35], [36], [37], [38], [39], and [40]. Orthogonal series expansion is also used to approximate the conditional probability density as in [41]. Also, the idea of generating a finite set of moments to replace the infinite set for the true density has been investigated in [44]. A more detailed account and discussion of the above mentioned techniques is given by the author in [61].

With all the above mentioned approaches for developing suboptimal finite dimensional filters, still the task of theoretical assessment of such filters in the sense of providing a measure of how far a suboptimal filter is from being a truly optimal has remained very hard to achieve.

It inherits the very same practical difficult- of the optimal filter - infinite dimensionality - that one is trying to avoid. Therefore, the support of any such schemes has to rely heavily on computer simulation and for that same reason not a single scheme can be claimed always superior. For example, in

[11], the truncated second filter, the Gaussian second order filter, the modified second order filter (M2-F), the extended Kalman filter (EKF), and the linearized filter were considered in numerical simulation. The linearized filter had the poorest performance but no conclusion was evident about which one of the other filters is superior. The EKF was favored for its relative structure simplicity in comparison to the other filters. Therefore, the final judgement is left to experience and the special case at hand. Consequently, the development of new practical filters will add to the list of contributors.

The main theme of this chapter is to consider the nonlinear filtering problem from a different approach. The approach taken here is to consider the problem as the combination of approximating the system description and solving the filtering problem for the approximate model. As a result some new schemes are developed. The problem formulation and the proposed solution are given next followed by some numerical results.

## II. Problem Formulation:

Consider the general nonlinear dynamical system whose state $x(t)$ evolves in time according to the following differential equation,

$$dx(t) = [A(t)\,x(t) + f(x(t),t)]\,dt + Q^{1/2}(t)\,dW(t) \quad (1)$$

$$x(t_o) = x_o \ , \quad t \geq t_o$$

where

$x(t) \in R^n$ is an 'n' dimensional state vector.

$A(t)$ is an 'nxn' real matrix.

$f(x(t),t)$ is an 'n' dimensional vector valued real function.

$x_o \in R^n$ is an 'n' dimensional Gaussian random vector (GRV) with

$$E\{x_o\} = \overline{x}_o \ ^{*} \quad (2)$$

and

$$Cov(x_o,x_o) \triangleq E\{(x_o - \overline{x}_o)(x_o - \overline{x}_o)'\} = P_o \ ^{\dagger} \quad (3)$$

$W(t) \in R^n$ is an 'n' dimensional Wiener process, and

$dW(t) = W(t+dt) - W(t)$. Therefore,

$$E\{dW(t)\} = o \text{ for all } t \geq t_o \quad (4)$$

and

$$Cov(dW(t),dW(t)) \triangleq E\{dW(t)\,dW'(t)\} = (Idt) \quad (5)$$

Where I is the (nxn) unit matrix.

$Q^{1/2}(t)$ is a real matrix, and

$Q(t) \triangleq Q^{1/2}(t)\,Q^{1/2'}(t)$ is a positive semidefinite (nxn) matrix.

* $E\{\cdot\}$ denotes the expected value of $(\cdot)$

† $Cov(\cdot,\cdot)$ denotes the covariance of $(\cdot)$.

Also, consider the observations process dy(t) to be given by

$$dy(t) = [C(t)\, x(t) + h(x(t),t)]\, dt + R^{\frac{1}{2}}(t)\, dv(t) \tag{6}$$

where

$dy(t) \in R^m$ is an 'm' dimensional observations vector.

$C(t)$ is an 'mxn' real matrix.

$h(x(t),t)$ is an 'm' dimensional vector valued real function.

$v(t) \in R^m$ is an 'm' dimensional Wiener process, and

$dv(t) = V(t + dt) - V(t)$. Therefore,

$$E\,\{dv(t)\} = o \text{ for all } t \geq t_0 \tag{7}$$

and

$$Cov(dv(t),\, dv(t)) \triangleq E\,\{dv(t)\;\; dv'(t)\} = (Idt) \tag{8}$$

$R^{\frac{1}{2}}(t)$ is a real matrix, and

$R(t) \triangleq R^{\frac{1}{2}}(t)\, R^{\frac{1}{2}'}(t)$ is a positive definite (nxn) matrix

We assume that $x_0$, $w(t)$, and $v(t)$ are all independent of each other for all values of $t \geq t_0$. Also, the assumption that equation (1) satisfies the conditions for existence and uniqueness of solution given in [3], [23], and [57] is being made. This means that our dynamical system (1) admits only one solution $x(t), t \geq t_0$ to be its state trajectory in the mean square sense. Furthermore, it is assumed that both $f(x(t),t)$ and $h(x(t),t)$ are continuous in $x(t)$.

As it is noticed from equations (1), and (6), the system structure is considered to be composed of two parts, a linear part plus a non-linear part. Furthermore, we assume that the system behavior is dominated by its linear part, That is to say,

$$\|f(x(t),t)\| < \|A(t)x(t)\| \tag{9}$$

and

$$\| h(x(t),t) \| < \| C(t)x(t) \| \tag{10}$$

where

$\|z\|$ is the norm of the vector z.

Equations (1) and (6) along with conditions (9) and (10) can be the original system's description, what is sometimes referred to as system with conebounded nonlinearities. Also, it can be a representation obtained by linearization of a nonlinear system, where $f(x(t),t)$ and $h(x(t),t)$ represent second and higher order terms. In this case conditions (9) and (10) are valid as long as the system state $x(t)$ remains within a small neighborhood of the nominal (linearizing) trajectory.

Accordingly, conditions (9), and (10) suggest that for a good guess of the system state $x^*(t)$ the following approximate equations for the dynamics and observations can be written as

$$dx_1(t) = [A(t)\, x_1(t) + f(x^*(t),t)]\, dt + Q^{\frac{1}{2}}(t)\, dw(t) \tag{11}$$

$$dy(t) = [C(t)\, x_1(t) + h(x^*(t),t)]\, dt + R^{\frac{1}{2}}(t)\, dv(t) \tag{12}$$

By virtue of continuity of the nonlinearities in $x(t)$, we should note the following. As $x^*(t)$ approaches $x_1(t)$, the approximate description given in (11), and (12) approaches the true description in (1), and (6). In fact, the following equation

$$dx_1(t) = [A(t)x_1(t) + f(x_1(t),t)]\, dt + Q^{\frac{1}{2}}(t)\, dw(t),$$
$$x(t_0) = X_0,\ t \geq t_0 \tag{13}$$

and equation (1) have the same solution both in the mean square sense and with probability one.

Thus follows, the filtering problem of the system (1), (6) can be considered as a unification of model approximation and state estimation of the approximate model. In other words, first we approximate the system description by finding a suitable $x^*(t)$. Then, solve the optimal filtering problem of the approximate model. The optimal filtering is basically to seek the minimum mean square error estimate of the state $x(t)$ based on the available observations, $Y_t=[y(s),\ t_o\leq s\leq t]$.

Generally, according to theorem (6.6) of [3] pp. 184 and its specialization to linear systems, theorem (7.3) pp. 219 of the same reference, the optimal filter imitates the dynamics of the system and is linearly driven by the net observations. Therefore, guided by these results, we will seek the optimal filter for the system in (11) and (12) as a linear dynamic system driven linearly by the net observations. The optimality of the filter is in the sense of achieving minimum mean square error.

so, if we define the estimation error $e_1(t)$ as

$$e_1(t)\triangleq x_1(t) - \hat{x}_1(t) \tag{14}$$

and the covariance matrix $P(t)$ as

$$P(t)\triangleq E\{(e_1(t) - \bar{e}(t))(e_1(t) - \bar{e}_1(t))'\} \tag{15}$$

Where $\hat{x}_1(t)$ is an estimate of $x_1(t)$ based on $Y_t$, and

$$\bar{e}_1(t) \triangleq E\ \{e_1(t)\} \tag{16}$$

then,

$$\begin{aligned} J(e_1(t)) &= tr\ (E\{e_1(t)e'_1(t)\}) \\ &= tr(P(t)) + tr(\bar{e}_1(t)\ \bar{e}_1'(t)) \end{aligned} \tag{17}$$

is to be minimized.

## III. Proposed Solutions:

### A. Derivation of The (El-F) Filter:

According to the approximate model in equations (11) and (12), the minimum variance unbiased estimate $\hat{x}_1(t)$ is given by a Kalman filter which has the following expression

$$\left.\begin{aligned}
d\hat{x}_1(t) &= [A(t)\hat{x}_1(t) + f(x^*(t),t)]\,dt + K(t)[dy(t) \\
&\quad - C(t)\hat{x}_1(t)dt - h(x^*(t),t)dt] \\
\hat{x}_1(t_o) &= \overline{X}_o \\
K(t) &= P(t)C'(t)R^{-1}(t) \\
dP(t) &= [A(t)P(t) + P(t)A'(t) - P(t)C'(t)R^{-1}(t)C(t)P(t) \\
&\quad + Q(t)]\,dt \\
P(t_o) &= P_o
\end{aligned}\right\} \quad (18)$$

A well known property of the Kalman filter is that $\hat{x}_1(t)$ is the conditional expectation of $x_1(t)$ given the measurements $Y_t$, i.e.

$$\hat{x}_1(t) = E_{Y_t}\{x_1(t)\}$$

According to the argument following equations (11) and (12), $x^*(t)$ is required to provide the optimal solution of the following minimization problem.

$$\min_{x^*(t)} J(x^*(t)) = E_{Y_t} \left\{(x_1(t) - x^*(t))'(x_1(t) - x^*(t))\right\} \tag{15}$$

then for every $t \geq t_0$ setting $\partial J(x^*(t))/\partial x^*(t) = 0$ we get

$$x^*(t) = E_{Y_t} \left\{ x_1(t)\right\} = \hat{x}_1(t) \tag{20}$$

Therefore, combining the results of equations (18), and (20) we get the following filter, to be denoted as the (El-F) filter, namely,

$$d\hat{x}(t) = \left[A(t)\hat{x}(t) + f(\hat{x}(t),t)\right]dt + K(t)\left[dy(t) - C(t)\hat{x}(t)\,dt - h(\hat{x}(t),t)\,dt\right], \quad \hat{x}(t_0) = \bar{x}_0 \tag{21}$$

$$K(t) = P(t)C'(t)R^{-1}(t) \tag{22}$$

$$dP(t) = \left[A(t)P(t) + P(t)A'(t) - P(t)C'(t)R^{-1}(t)C(t)P(t) + Q(t)\right]dt, \quad P(t_0) = P_0 \tag{23}$$

It is straightforward to recognize that in case of a linear system, i.e. $f(x(t),t)$ and $h(x(t),t)$ are identically zero or only functions of time, equations (21), (22) and (23) reduce to the well known Kalman filter.

The extended Kalman filter (EKF), [3]
(1) and (6) is given by the following equations.

$$d\hat{x}(t) = [A(t)\hat{x}(t)+f(\hat{x}(t),t)]dt + K(t)\,[dy(t) - C(t)\,\hat{x}\,(t)\,dt - h(\hat{x}(t),t)dt],\ \hat{x}(t_o) = \bar{x}_o \tag{24}$$

$$K(t) = P(t)\,[C(t) + h_x(\hat{x}(t),t)]'\,R^{-1}(t) \tag{25}$$

$$dP(t) = \{[A(t) + f_x(\hat{x}(t),t)]\,P(t) + P(t)\,[A(t) + f_x(\hat{x}(t),t)]' - P(t)(C(t) + h_x(\hat{x}(t),t))'\,R^{-1}(t)(C(t) + h_x(\hat{x}(t),t)).\,P(t) + Q(t)\}\,dt,\ P(t_o) = P_o \tag{26}$$

where

$$f_x\,(\hat{x}(t),\,t) = \left.\frac{\partial f(x(t),t)}{\partial x(t)}\right|_{x(t) = \hat{x}(t)}$$

and

$$h_x\,(\hat{x}(t),t) = \left.\frac{\partial h(x(t),t)}{\partial x(t)}\right|_{x(t) = \hat{x}(t)}$$

The (El-F) bears a close relationship with the extended Kalman filter (EKF). The equations for the state estimate of both the (El-F) and the (EKF), equations (21) and (24), have the same structure. While the equations for the gain and covariance of the (El-F), equations (22) and (23), are different from those for the (EKF), equations (25) and (26). Equations (22) and (23) are no longer state estimate dependent. Thus, unlike the (EKF), the gain and covariance for the (El-F) can be processed off line and prior to receiving the observations like the Kalman filter (KF). Therefore, the El-F will be of advantage over the EKF when on line computations of the gain and covariance are not affordable due to capacity limitations of on line computers. This is usually the case of airborn and spaceborn computers.

Furthermore, while the (EKF) has to be strictly interpreted in the Itô sense, [62]. it is not the case with the (El-F). This is so because the gain K(t) as given by equation (22) is not estimate dependent.

## B. Numerical Experiment for (El-F):

The Van der Pol oscillator is chosen to compare the following filters, (El-F), (KF), and (EKF). The Van der Pol oscillator is characterized by the following differential equation, [24].

$$\ddot{x}(t) - \varepsilon\dot{x}(t)(1 - x^2(t)) + x(t) = 0 \qquad (27)$$

which describes a dynamical system with state dependent damping coefficient equals $-\varepsilon(1-x^2(t))$ where $\varepsilon$ is a positive parameter. The damping in the system goes from negative to zero to positive values as the value of $x^2(t)$ changes from less than to greater than unity. The oscillator's response is characterized by a limit cycle in the $x(t)$, $\dot{x}(t)$ plane (the phase plane). The limit cycle approaches a circular shape as $\varepsilon$ becomes very small, it has a maximum value for $x(t)$ equals 2.0 irrespective of the value of $\varepsilon$. This type of oscillations occur in electronic tubes which exhibit also what is known as thermal noise. Denoting $x(t)$ as $x_1(t)$, and $\dot{x}(t)$ as $x_2(t)$, equation (23) can be rewritten in a state space formulation. Also, considering the existence of some noise forcing on the system, we get the following representation for the Van der Pol oscillator.

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ -\varepsilon x_1^2(t)\, x_2(t) \end{bmatrix} dt + Q^{\frac{1}{2}} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \qquad (28)$$

Also suppose that the following measurement is taken

$$dy(t) = \left[x_1(t) + x_1^3(t)\right] dt + R^{1/2} dv(t) \tag{29}$$

In (24) and (25) above $[W_1(t)\ W_2(t)]^T$ is considered to be a two dimensional Wiener process. Also, V(t) is a one dimensional Wiener process. R is a positive nonzero real value, and Q is a (2x2) matrix. The following values for noise statistics are considered.

| Case # | $Q_{11}$ | $Q_{12}$ | $Q_{22}$ | R | figures |
|---|---|---|---|---|---|
| Van der Pol 1 | 0.5 | 0.0 | 0.5 | 4.0 | 1 to 2 |
| Van der Pol 2 | 5.0 | 2.0 | 5.0 | 10.0 | 3 to 4 |

Also $\varepsilon$ is taken to be 0.2

In the figures, the following symbols are used.

XI $\equiv$ the $i^{\underline{th}}$ state, I = 1, 2

XIK $\equiv$ the estimate of the $i^{\underline{th}}$ state provided by the (K-F)

XIE $\equiv$ the estimate of the $i^{\underline{th}}$ state provided by the (E1-F)

XIEK $\equiv$ the estimate of the $i^{\underline{th}}$ state provided by the (EKF)

In both cases; as indicated by figures 1, 2, 3, and 4, both the (E1-F) and (EKF) provide very accurate tracking of the system's states while the (KF) provides crude estimates.

## FIGURES CAPTIONS

Figure 1. First state and estimate by Kalman, El, Extended Kalman filters.

Figure 2. Second state and estimates by Kalman, El, Extended Kalman filters.

Figure 3. First state and estimates by Kalman, El, Extended Kalman fitlers. .

Figure 4. Second state and estimates by Kalman, El, Extended Kalman filters.

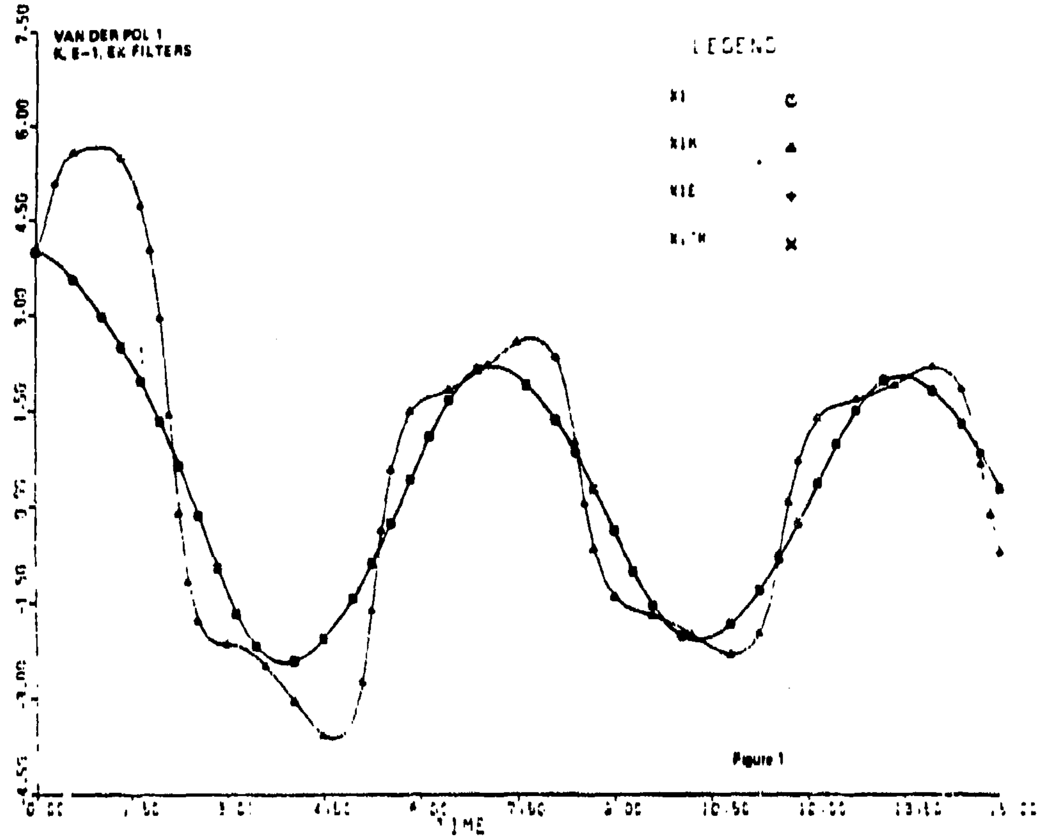


Figure 1

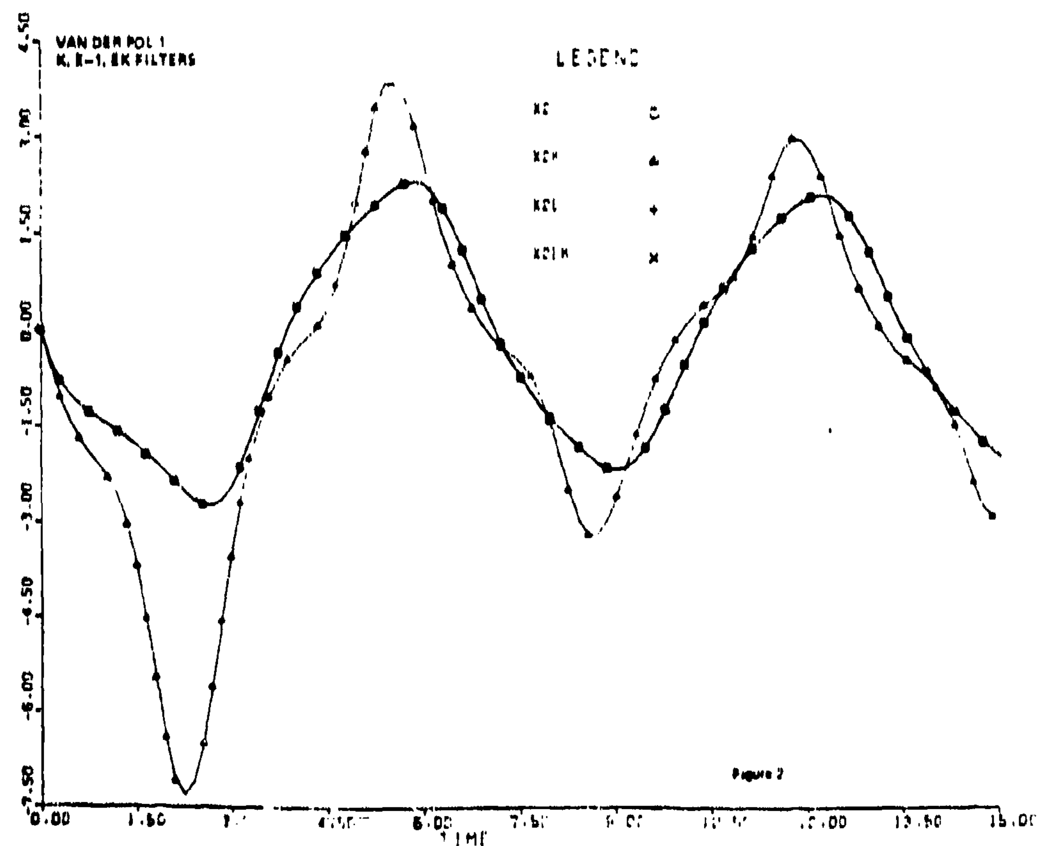


Figure 2

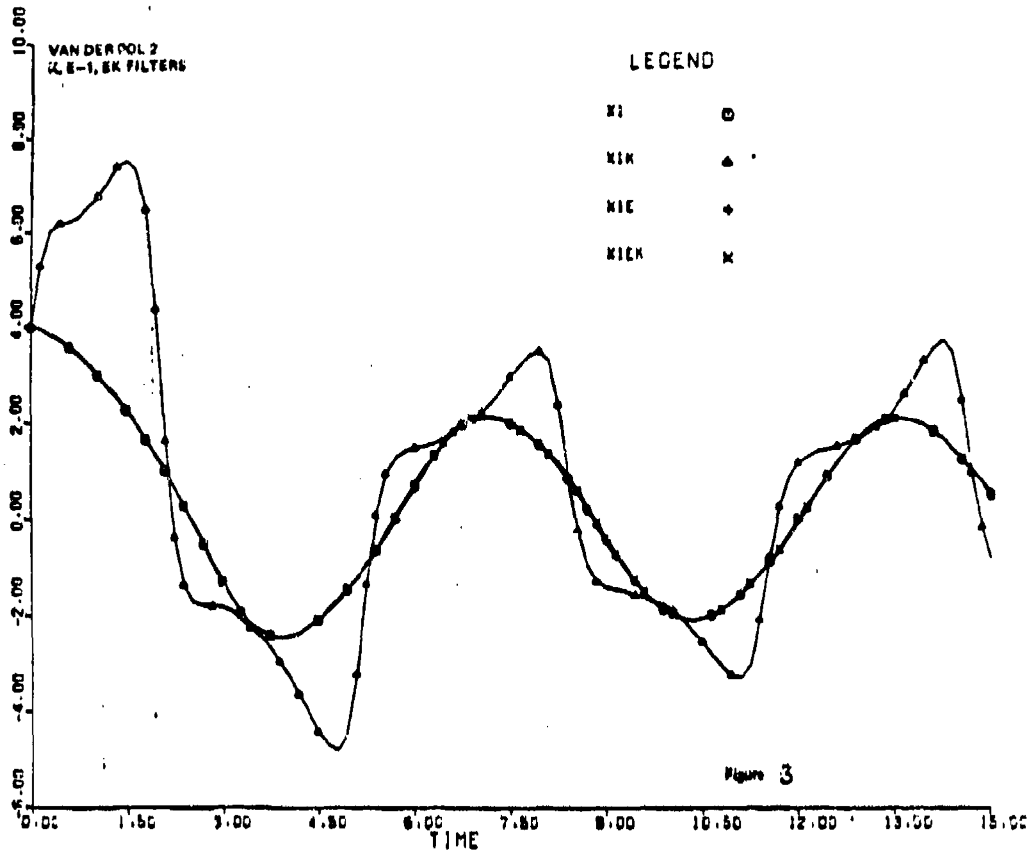


Figure 3

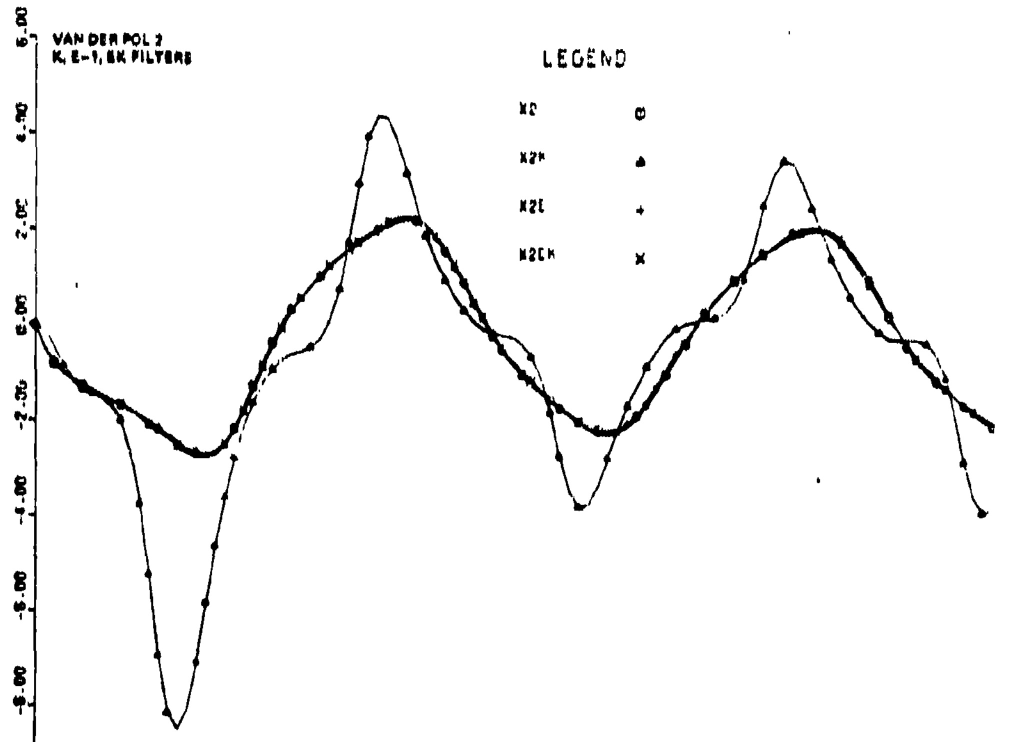


Figure 4

## C. Derivation of the E2-F and the E2N-F filters

We have

$$dx_1(t) = \left[A(t)x_1(t) + f(x^*(t),t)\right] dt + Q^{\frac{1}{2}}(t)\, dw(t) \tag{30}$$

$$dy(t) = \left[C(t)x_1(t) + h(x^*(t),t)\right] dt + R^{\frac{1}{2}}(t)\, dv(t) \tag{31}$$

as our approximate model for some given good guess of the system state $x^*(t)$.

Then we seek a filter which is a linear dynamic system, linearly driven by the available observations as follows

$$d\hat{x}_1(t) = \left[B(t)\hat{x}_1(t)\right] dt + K(t)\, dy(t) \tag{32}$$

where

B(t) is an 'nxn' matrix and K(t) is an 'nxm', the filter's gain matrix.

In order to evaluate the accuracy of this filter in estimating the state $x_1(t)$, we define the estimation error $e_1(t)$ as

$$e_1(t) \triangleq x_1(t) - \hat{x}_1(t) \tag{33}$$

Therefore from (30), (31), and (32) we get

$$\begin{aligned} de_1(t) = &\Big[(A(t) - K(t)\,C(t) - B(t))\,x_1(t) + B(t)\,e_1(t) \\ &+ f(x^*(t),t) - K(t)\,h(x^*(t),t)\Big]\,dt \\ &+ Q^{1/2}(t)\,dw(t) - K(t)\,R^{1/2}(t)\,dv(t), \\ e_1(t_0) = &\ x_0 - \hat{x}_1(t_0) \end{aligned} \tag{34}$$

It is desirable to have the estimation error independent of the state. In this case large state variables can be estimated as accurate as small state variables.

Therefore, we may choose

$$B(t) = A(t) - K(t)\,C(t). \tag{35}$$

Hence, the dependence of the estimation error on the state is eliminated. Also, the initial minimum variance estimate is the mean of the initial state $x_0$.

Therefore,

$$\hat{x}_1(t_0) = \overline{x}_0 \tag{36}$$

Hence, equation (34) reduces to

$$
\begin{aligned}
de_1(t) = \Big[(A(t) - K(t)\, C(t))\, e_1(t) + f(x^*(t),t) \\
- K(t)\, h(x^*(t),t)\Big]\, dt + Q^{1/2}(t)\, dw(t) \\
- K(t)\, R^{1/2}(t)\, dv(t),\ e_1(t_o) = x_o - \bar{x}_o
\end{aligned} \tag{37}
$$

Accordingly, the equation for the mean value of the error $\bar{e}_1(t)$ is as follows.

$$
\begin{aligned}
d\bar{e}_1(t) = \Big[(A(t) - K(t)\, C(t))\, \bar{e}_1(t) + f(x^*(t),t) \\
- K(t)\, h(x^*(t),t)\Big]\, dt,\ \bar{e}_1(t_o) = o
\end{aligned} \tag{38}
$$

It is clear that equation (38) above, due to the term $\big[f(x^*(t),t) - K(t)\, h(x^*(t),t)\big]$, will have a non zero solution, i.e.

$$
\bar{e}_1(t) \equiv E\{e_1(t)\} \neq o \tag{39}
$$

Hence our estimate is biased unless the term $\big[f(x^*(t),t) - K(t)\, h(x^*(t),t)\big]$ is identically equal to zero for all values of $t \geq t_o$.

From equations (37) and (38) above, we have

$$
\begin{aligned}
de_1(t) - d\bar{e}_1(t) = \big[A(t) - K(t)\, C(t)\big](e_1(t) - \bar{e}_1(t))\, dt \\
+ Q^{1/2}(t)\, dw(t) - K(t)\, R^{1/2}(t)\, dv(t), \\
e_1(t_o) - \bar{e}_1(t_o) = x_o - \bar{x}_o
\end{aligned} \tag{40}
$$

By definition the covariance matrix P(t) is

$$P(t) = E\{(e_1(t) - \bar{e}_1(t))\,(e_1(t) - \bar{e}_1(t))'\} \tag{41}$$

Therefore, straight forward mathematical manipulations show that P(t) is given by the following differential equation.

$$\begin{aligned} dP(t) = \Big[&(A(t) - K(t)\,C(t))\,P(t) + P(t)\,(A(t) - K(t)\,C(t))' \\ &+ Q(t) + K(t)\,R(t)\,K'(t)\Big]\,dt,\quad P(t_0) = P_0 \end{aligned} \tag{42}$$

Next, we seek the gain K(s), $t_0 \le s \le t$ that will provide the minimum mean square error. Therefore we formulate the following optimization problem

$$\begin{aligned} \min_{\substack{K(s)\\ t_0 \le s \le t}}\ tr(P(t)) + \int_{t_0}^{t} &\big[f(x^*(s),s) - K(s)\,h(x^*(s),s)\big]'\,\big[f(x^*(s),s) \\ &- K(s)\,h(x^*(s),s)\big]\,ds \end{aligned} \tag{43}$$

Subject to the constraint given by (42).

This can be rewritten as the following minimization problem,

$$\begin{aligned} \min_{\substack{K(s)\\ t_0 \le s \le t}} \int_{t_0}^{t} tr\Big(&\big[A(s) - K(s)\,C(s)\big]\,P(s) + P(s)\,\big[A(s) - K(s)\,C(s)\big]' \\ &+ K(s)\,R(s)\,K'(s) + \big[f(x^*(s),s) - K(s)\,h(x^*(s),s)\big]. \\ &\big[f(x^*(s),s) - K(s)\,h(x^*(s),s)\big]'\Big)\,ds \end{aligned} \tag{44}$$

The integrand in (44) is a convex quadratic in K(t). According to the theory of calculus of variations, [19], the minimizing K(s), $t_0$ s t is given as the solution of the Euler's equation which reduces to a simple algebraic equation in the present case, namely

$$\frac{\partial}{\partial K(t)} \operatorname{tr}([A(t) - K(t)C(t)]P(t) + P(t)[A(t) - K(t)C(t)]' + K(t)R(t)K'(t) + [f(x^*(t),t) - K(t)h(x^*(t),t)]\,[f(x^*(t),t) - K(t)h(x^*(t),t)]') = 0 \tag{45}$$

Using the concept of gradient matrices and the formulae developed in [52], we get

$$\frac{\partial}{\partial K(t)} \operatorname{tr}(K(t)C(t)P(t)) = P(t)C'(t) \tag{46}$$

$$\frac{\partial}{\partial K(t)} \operatorname{tr}(P(t)C'(t)K'(t)) = P(t)C'(t) \tag{47}$$

$$\frac{\partial}{\partial K(t)} \operatorname{tr}(K(t)R(t)K'(t)) = 2K(t)R(t) \tag{48}$$

And,

$$\frac{\partial}{\partial K(t)} \operatorname{tr}([f(x^*(t),t) - K(t)h(x^*(t),t)]\,[f(x^*(t),t) - K(t)h(x^*(t),t)]') = -2f(x^*(t),t)h'(x^*(t),t) + 2K(t)h(x^*(t),t)\;h'(x^*(t),t) \tag{49}$$

Substituting (46), (47), (48), and (49) in (45) above, the optimal gain is found to satisfy the following equation.

$$K(t)\left[R(t) + h(x^*(t),t)\,h'(x^*(t),t)\right] = P(t)C'(t) + f(x^*(t),t)h'(x^*(t),t) \tag{50}$$

Therefore, the solution to the filtering problem of the approximate model is given by

$$\left.\begin{aligned}
&d\hat{x}_1(t) = A(t)\,\hat{x}_1(t)\,dt + K(t)\left[dy(t) - C(t)\,\hat{x}_1(t)\,dt\right] \\
&\hat{x}_1(t_0) = \bar{x}_0 \\
&K(t) = \left[P(t)C'(t) + f(x^*(t),t)\,h'(x^*(t),t)\right]\cdot \\
&\qquad\left[R(t) + h(x^*(t),t)\,h'(x^*(t),t)\right]^{-1} \\
&dP(t) = \left[(A(t) - K(t)\,C(t))\,P(t) + P(t)\,(A(t) - K(t)\,C(t))' \right.\\
&\qquad\left. + Q(t) + K(t)\,R(t)\,K'(t)\right] dt,\; P(t_0) = P_0
\end{aligned}\right\} \tag{51}$$

It is clear that the inverse in the gain equation (50) exists because $R(t)$ is a positive definite matrix and $h(x_1^*(t),t)\,h'(x^*(t),t)$ is always a positive semidefinite matrix.

Although the bias term $[f(x^*(t),t) - K(t)\,h(x^*(t),t)]$ has been minimized, by choosing the gain $K(t)$ according to (50), it is not identically zero. The bias can be eliminated by modifying the state estimate equation such that the filter will be as follows.

$$
\left.
\begin{aligned}
d\hat{x}_1(t) &= \left[A(t)\,\hat{x}_1(t) + f(x^*(t),t)\right] dt \\
&\quad + K(t)\left[dy(t) - (C(t)\,\hat{x}_1(t) + h(x^*(t),t))\,dt\right] \\
\hat{x}_1(t_o) &= \overline{X}_o \\
K(t) &= \left[P(t)\,C'(t) + f(x^*(t),t)\,h'(x^*(t),t)\right] \cdot \\
&\quad \left[R(t) + h(x^*(t),t)\,h'(x^*(t),t)\right]^{-1} \\
dP(t) &= \left[(A(t) - K(t)\,C(t))\,P(t)\right. \\
&\quad + P(t)\,(A(t) - K(t)\,C(t)\,) \\
&\quad \left. + Q(t) + K(t)\,R(t)\,K'(t)\right] dt,\; P(t_o) = P_o
\end{aligned}
\right\} \quad (52)
$$

Next, the guessed nominal trajectory $x^*(t)$ is to be updated optimally in a sense to drive it as close as possible to $x_1(t)$. Hence, the following minimization problem is formulated.

$$\min_{x^*(t)} \; J(x^*(t)) = E_{Y_t}\,\{(x_1(t) - x^*(t))'(x_1(t) - x^*(t))\} \qquad (53)$$

Then for every $t \geq t_o$ setting $\partial J(x^*(t))/\partial x^*(t) = o$ we get

$$x^*(t) = E_{Y_t}\,\{x_1(t)\} = \hat{x}_1(t) \qquad (54)$$

Now, by combining the results in (51) and (54) we obtain the (E2-F) filter as follows.

$$
\left.
\begin{aligned}
& d\hat{x}_1(t) = A(t)\,\hat{x}_1(t)\,dt + K(t)\left[dy(t) - C(t)\,\hat{x}_1(t)\right]dt \\
& \hat{x}_1(t_0) = \bar{x}_0 \\
& K(t) = \left[P(t)\,C'(t) + f(\hat{x}_1(t),t)\,h'(\hat{x}_1(t),t)\right] \\
& \qquad \left[R(t) + h(\hat{x}_1(t),t)\,h'(\hat{x}_1(t),t)\right]^{-1} \\
& dP(t) = \Big[\big(A(t) - K(t)\,C(t)\big)\,P(t) + P(t)\,\big(A(t) - K(t)\,C(t)\big)' \\
& \qquad + K(t)\,R(t)\,K'(t) + Q(t)\Big]\,dt \\
& P(t_0) = P_0
\end{aligned}
\right\} \quad (55)
$$

And, by combining the results in (52) and (54) we obtain the (E2N-F) filter as follows

$$
\left.
\begin{aligned}
d\hat{x}_1(t) &= \left[A(t)\,\hat{x}_1(t) + f(\hat{x}_1(t),t)\right] dt \\
&\quad + K(t)\left[dy(t) - (C(t)\,\hat{x}_1(t) \right. \\
&\quad \left. + h(\hat{x}_1(t),t))\,dt\right] \\
\hat{x}_1(t_0) &= \overline{x}_0 \;. \\
K(t) &= \left[P(t)\,C'(t) + f(\hat{x}_1(t),t)\,h'(\hat{x}_1(t),t)\right] \cdot \\
&\quad \left[R(t) + h(\hat{x}_1(t),t)\,h'(\hat{x}_1(t),t)\right]^{-1} \\
dP(t) &= \left[(A(t) - K(t)\,C(t))\,P(t) + P(t)\,(A(t) - K(t)\,C(t))' \right. \\
&\quad \left. + K(t)\,R(t)\,K'(t) + Q(t)\right] dt \\
P(t_0) &= P_0
\end{aligned}
\right\} \quad (56)
$$

Few points should be mentioned in commenting on the results given by the equations in (55) and (56). It is easy to recognize that both the (E2-F) and the (E2N-F) will reduce to the standard Kalman filter (KF) when there is no nonlinearities in the system structure. The (E2-F) has a linear structure for the state estimate equation. But, the gain matrix $K(t)$ and the covariance matrix $P(t)$ for both filters in (55) and (56) are state estimate dependent, a common feature in many of the suboptimal nonlinear filters. The results indicate that the measurement nonlinearities have an effect on the filter gain similar to adding to the measurement noise by increasing its covariance. On the other hand both the dynamics and the measurements nonlinearities have a combined effect similar to $P(t)\,C'(t)$. If there is no measurements nonlinearities ($h(x(t),t) \equiv 0$) then the (E2-F) will reduce to the standard Kalman filter (KF) without compensating for the dynamics nonlinearities, while (E2N-F) will reduce to the (E1-F) given by equations (21)(22) and(23)

## D. NUMERICAL EXPERIMENTS FOR (E2-F), and E2N-F):

As before the Van der Pol Oscillator is chosen to compare the following filters. (E2-F), and (KF) in one experiment; and (E2N-F), (EKF), and (M2-F) in the second experiment.

The following values for noise statistics are considered.

| Case # | $Q_{11}$ | $Q_{12}$ | $Q_{22}$ | R | Figures |
|---|---|---|---|---|---|
| Van der Pol 1 | 0.5 | 0.0 | 0.5 | 4.0 | 5 through 8 |
| Van der Pol 2 | 5.0 | 0.0 | 5.0 | 10.0 | 9 through 12 |
| Van der Pol 3 | 10.0 | 0.0 | 10.0 | 20.0 | 13 through 16 |

As before $\epsilon$ is taken to be 0.2

the following symbols are used.

XI ≡ the i th state, I = 1, 2

XIK ≡ the estimate of the i th state provided by the (K-F)

XIE ≡ " " " " " " " " the (E2-F)

XIEN ≡ " " " " " " " " the (E2N-F)

XIEK ≡ " " " " " " " "the (EKF)

XIM ≡ " " " " " " " the (M2-F)

## Figure Captions

Figure 5. First state and estimates by Kalman and E2 Filters.

Figure 6. Second state and estimates by Kalman and E2 Filters.

Figure 7. First state and estimates by Extended Kalman, E2N, and modified second order filters.

Figure 8. Second state and estimates by Extended Kalman, E2N, and modified second order filters.

Figure 9. First state and estimates by Kalman and E2 Filters.

Figure 10. Second state and estimates by Kalman and E2 Filters.

Figure 11. First state and estimates by Extended Kalman, E2N, and modified second order filters.

Figure 12. Second state and estimates by Extended Kalman, E2N, and modified second order filters.

Figure 13. First state and estimates by Kalman and E2 Filters.

Figure 14. Second state and estimates by Kalman and E2 Filters.

Figure 15. First state and estimates by Extended Kalman, E2N, and modified second order filters.

Figure 16. Second state and estimates by Extended Kalman, E2N, and modified second order filters.

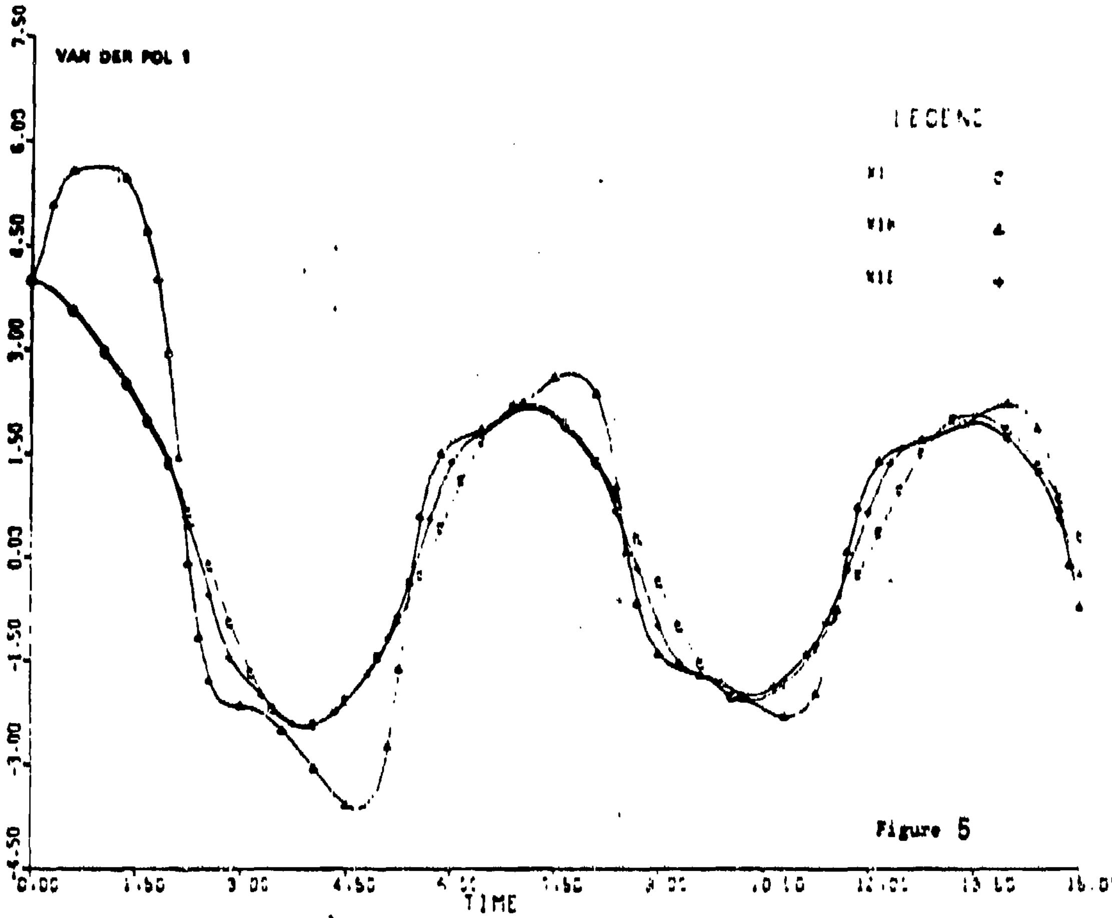


Figure 5

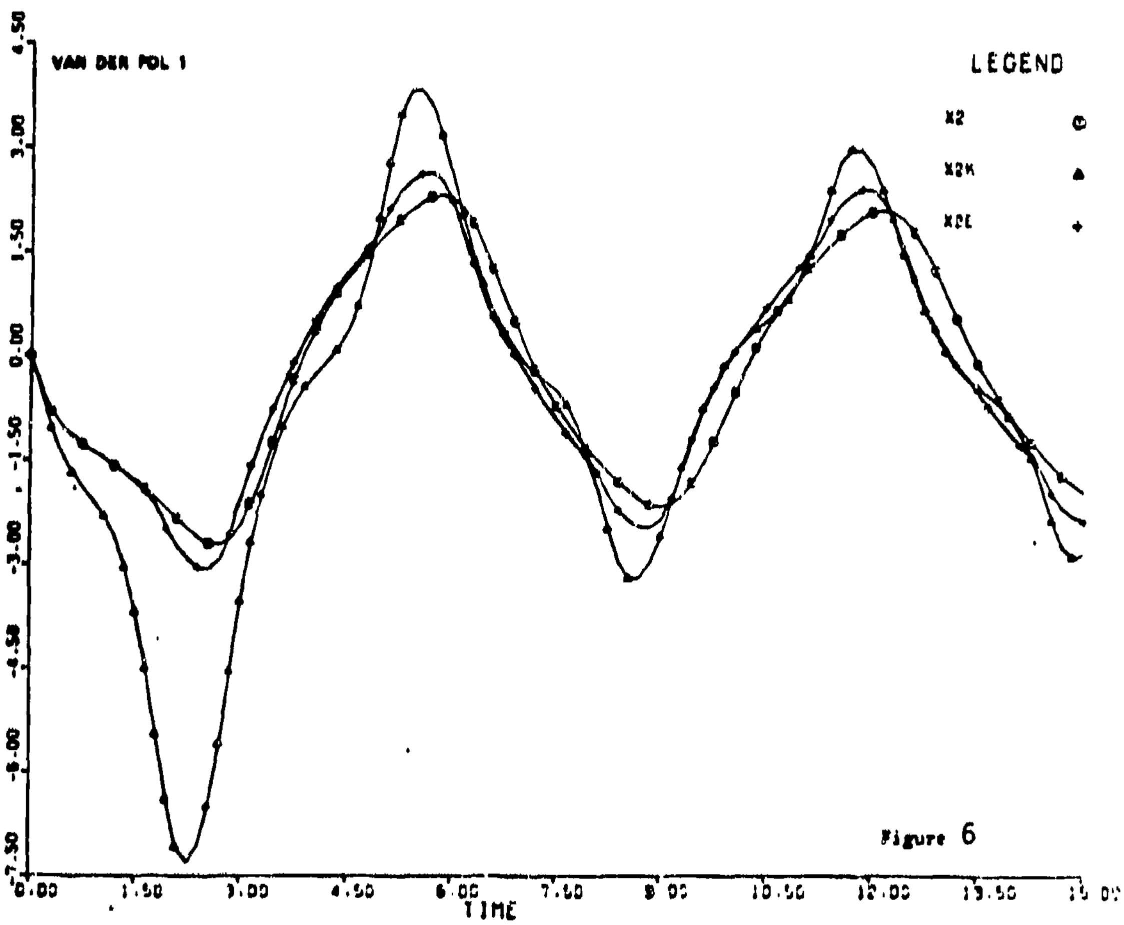


Figure 6

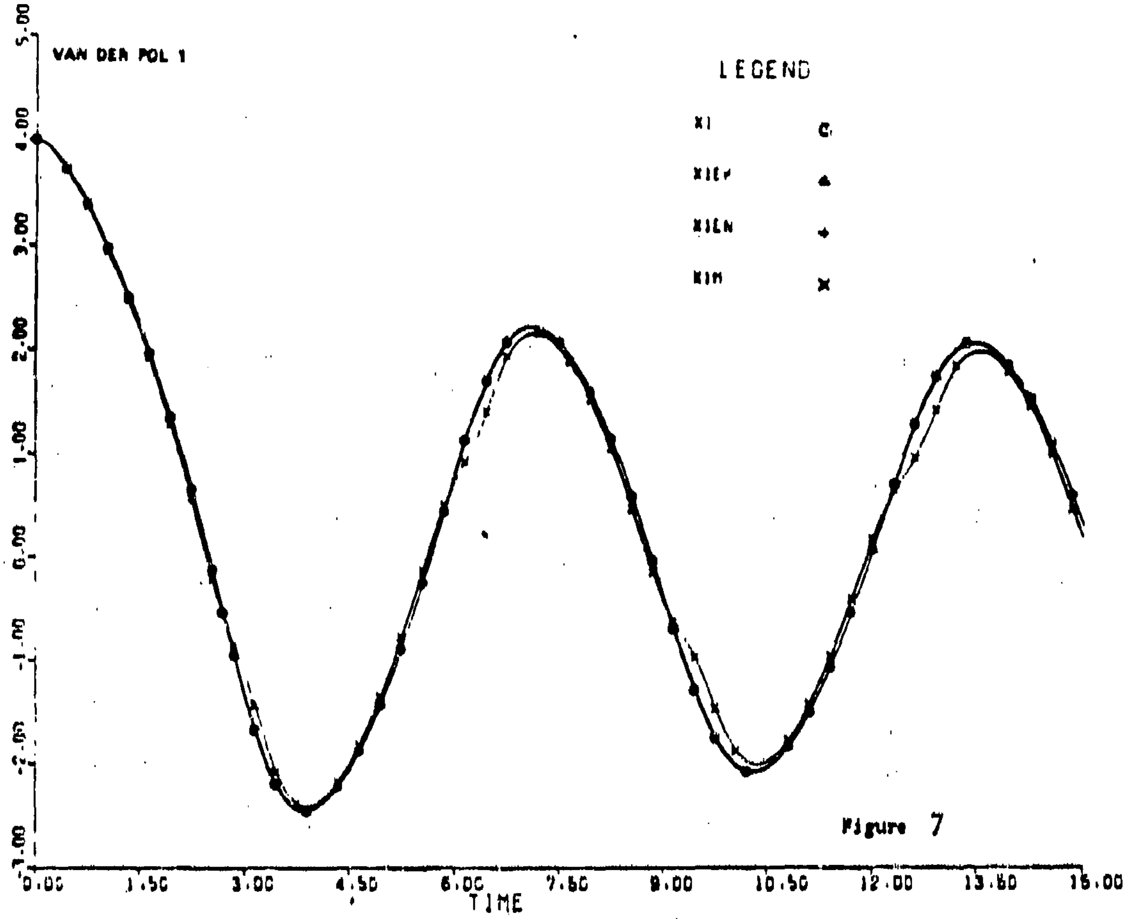


Figure 7

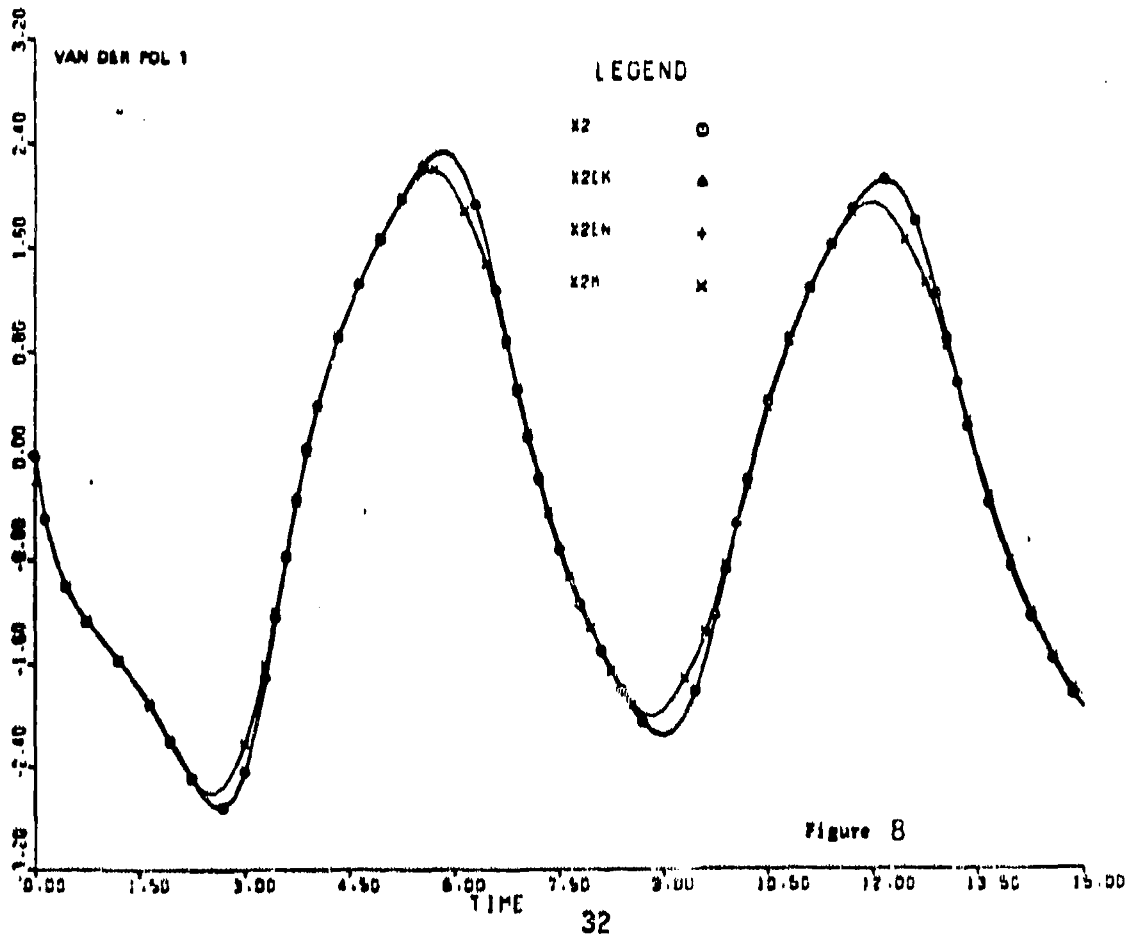


Figure 8

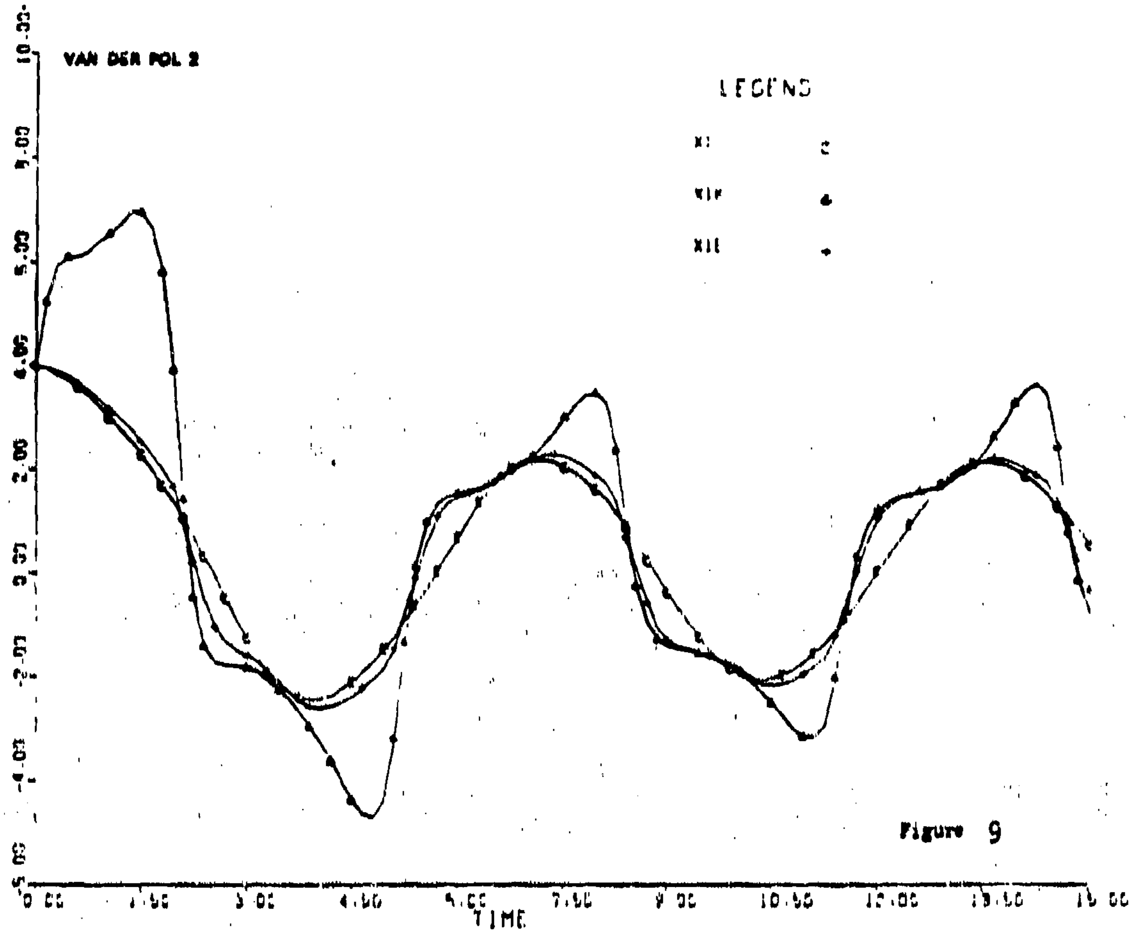


Figure 9

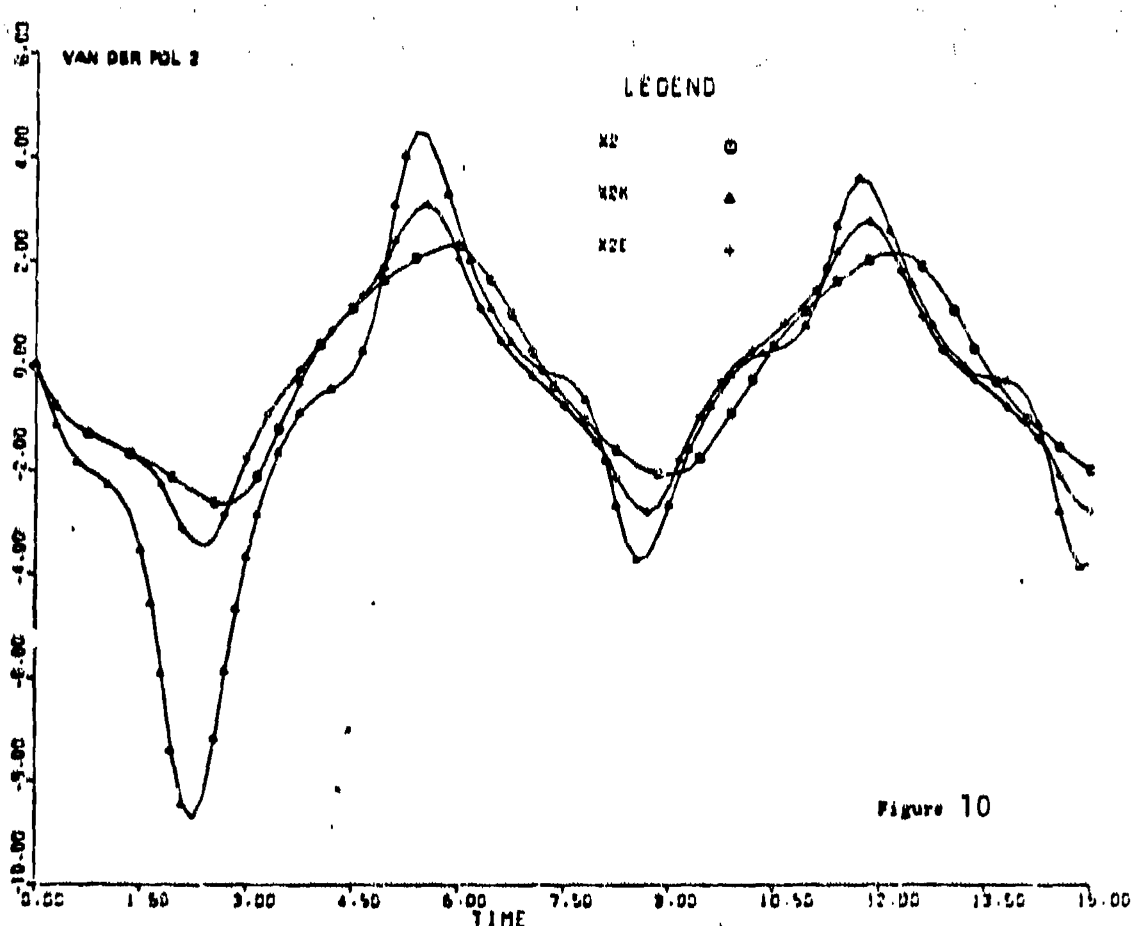


Figure 10

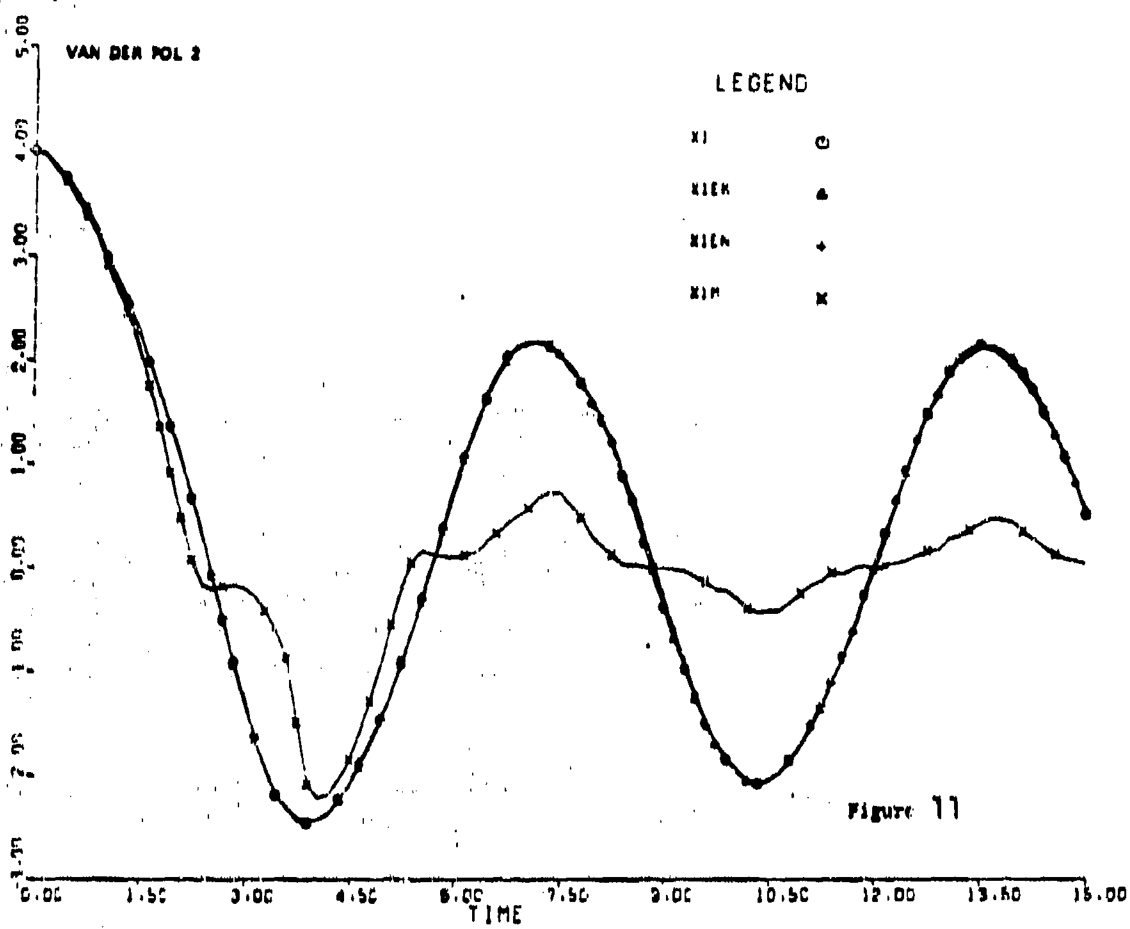


Figure 11

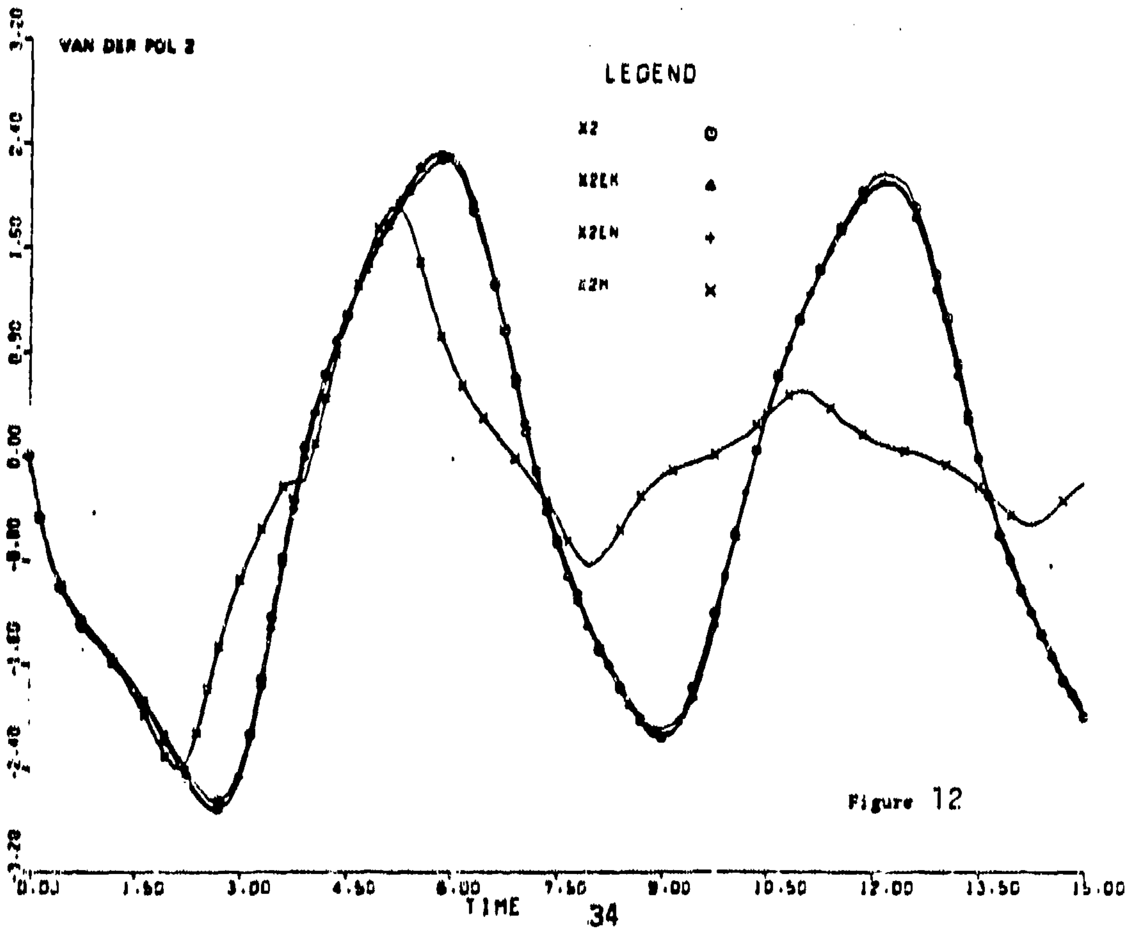


Figure 12

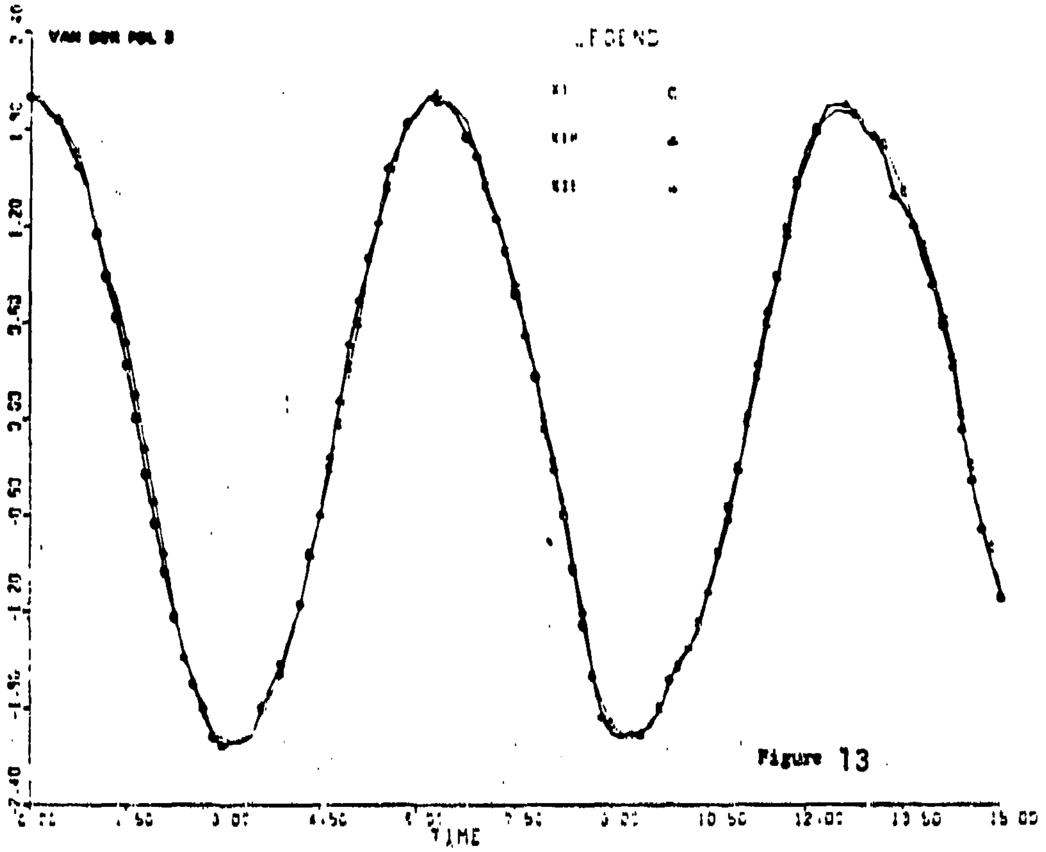


Figure 13

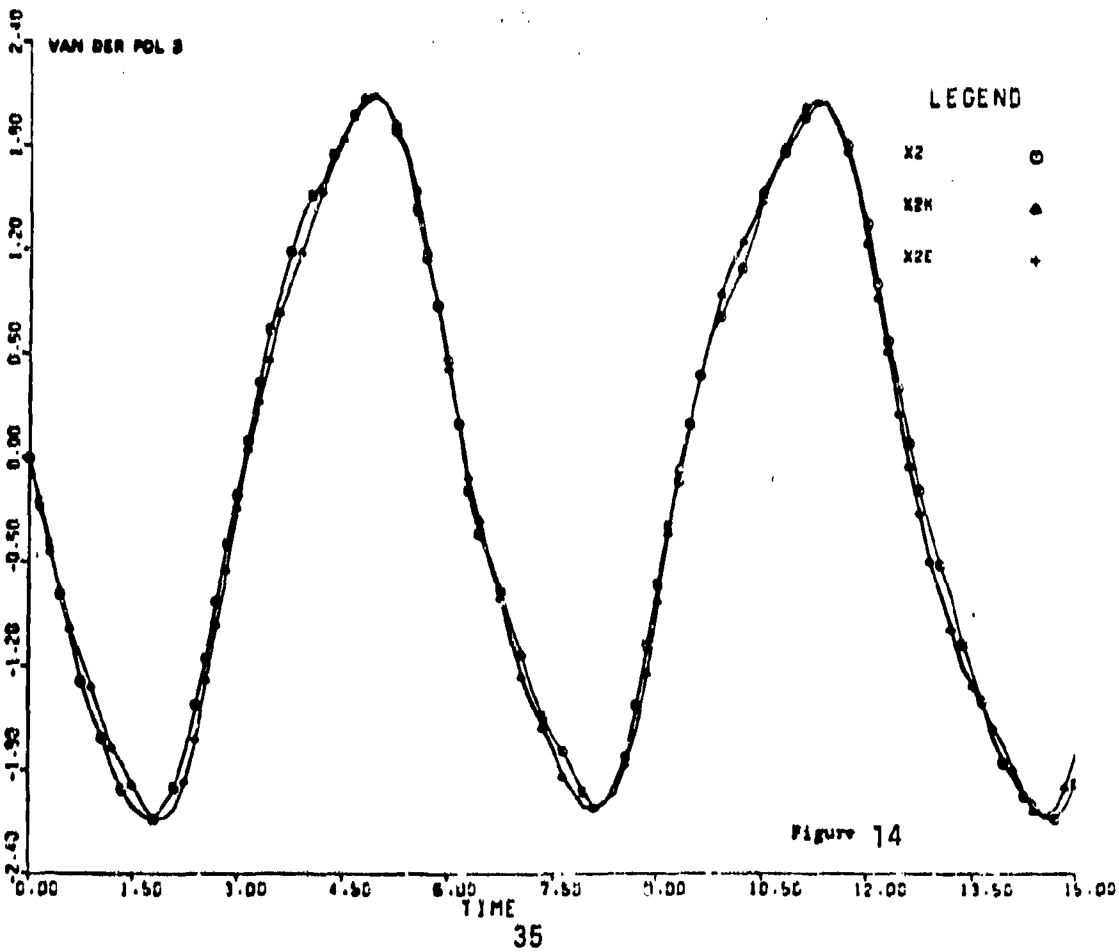


Figure 14

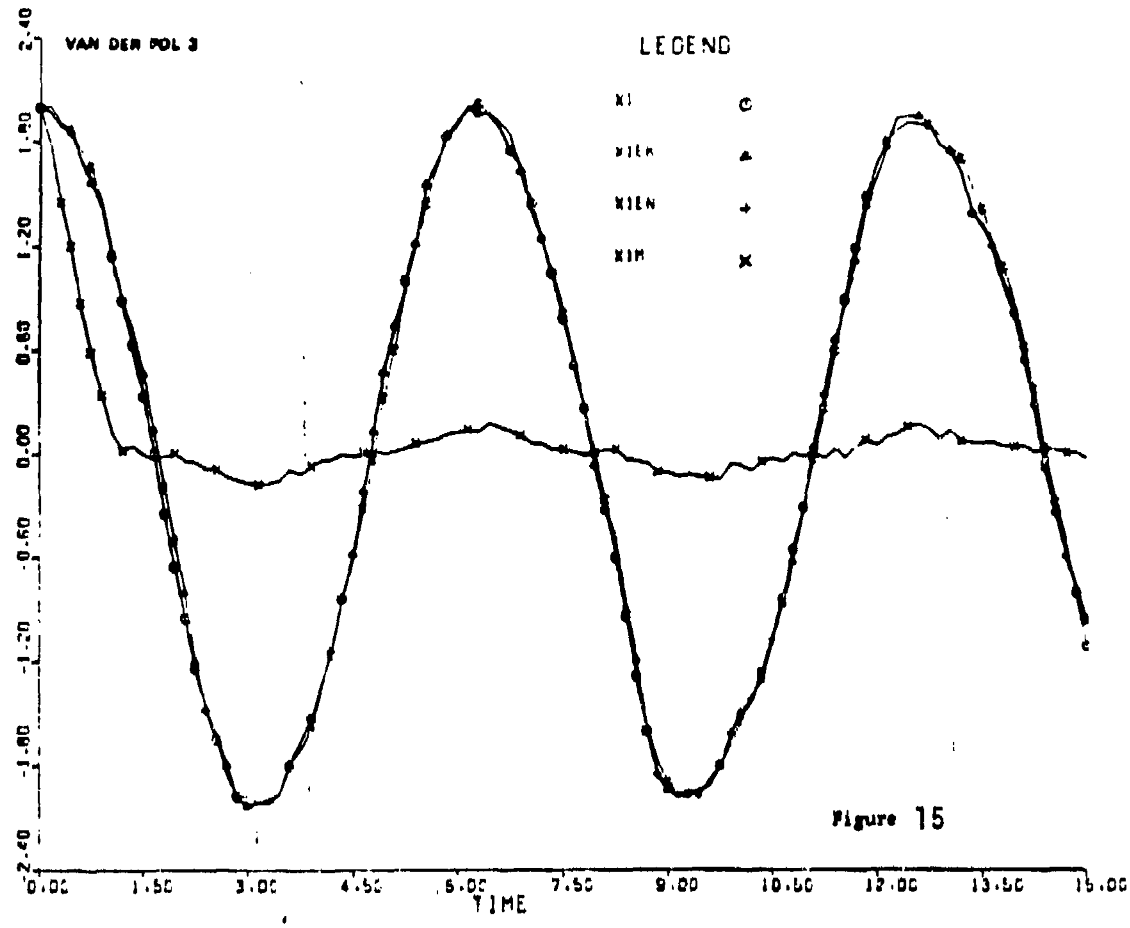


Figure 15

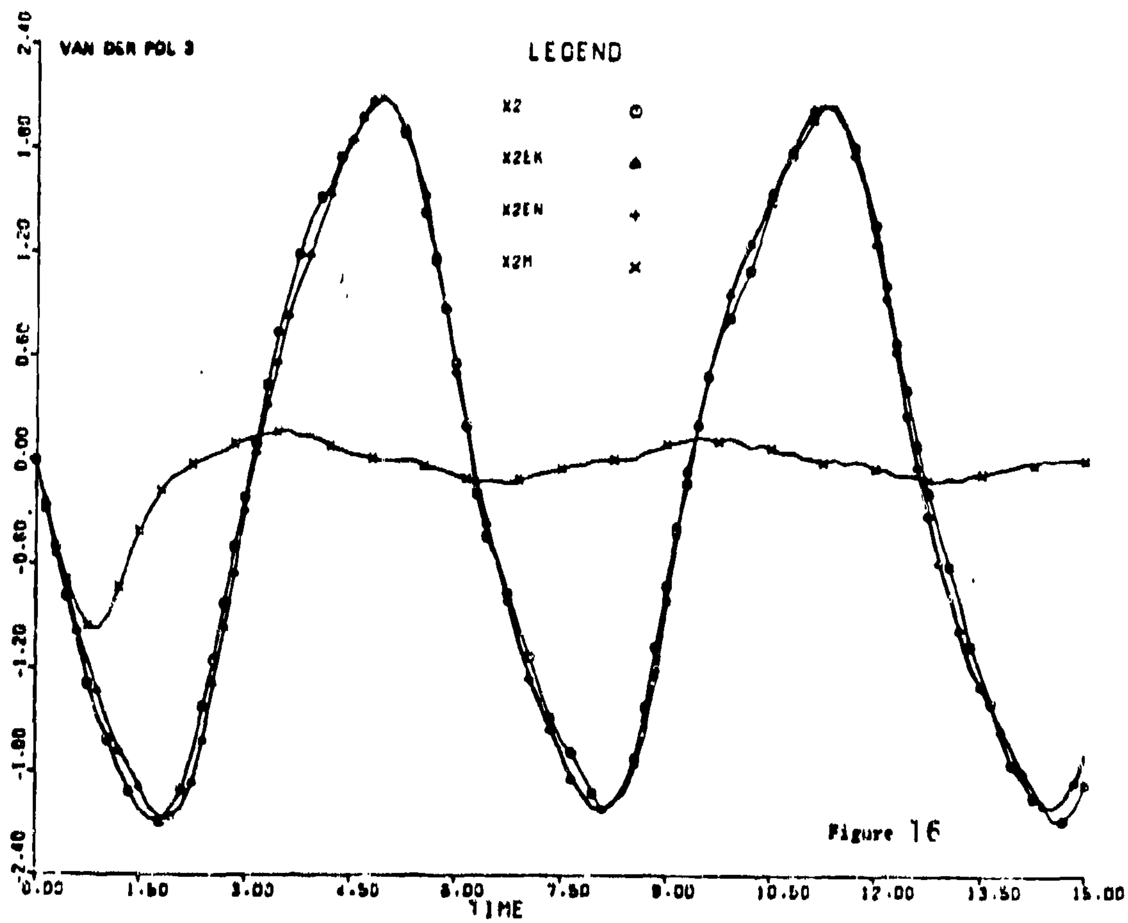


Figure 16

From the previously displayed sets of results the following can be pointed out. In the first two sets of results, Van der Pol 1 and Van der Pol 2 the (E2-F) provides a better tracking of the system states than the (K-F). This is evident from figures 5, 6, 9, and 10. It is clear that the (E2N-F) has accuracy similar to that of the (EKF) which is better than the (M2-F) as indicated by figures 7, 8, 11, and 12. In the third set of results, Van der Pol 3 figures 13 through 16 the noise level is high enough to cover the effect of the system nonlinearities. Therefore, all filters except the (M2-F) have similar performance. The (M2-F) is badly degraded and provides a crude estimate of the system state.

## IV. CONCLUSIONS:

In this document

~~The~~ the nonlinear filtering problem is treated using a new approach. The approach consists of unifying a system model approximation technique with the filtering solution based on the approximate model. As a result, several filters are developed.

The first filter (E1-F) structurally fits into the gap between the Kalman (KF) and the extended Kalman (EKF) filters. On one hand it enjoys the same computational facility enjoyed by the Kalman filter, namely, the off-line computations of its gain matrix. And on the other hand it provides state estimates on the same level of accuracy as provided by the extended Kalman filter. Therefore, in this sense the (E1-F) provides a missing link between (KF) and (EKF).

The other two filters are referred to as the (E2-F) and the (E2N-F). The state estimate provided by the (E2-F) has a structure like (KF) while that of the (E2N-F) has a structure like the (EKF). Both filters have new formula for the gain which provides further insight into the effects of the system nonlinearities. Specifically, measurements nonlinearities have the effect of increasing the measurements noise level. Moreover, the dynamics nonlinearities, and also the measurements nonlinearities have a combined effect similar to the $P(t)C'(t)$ term in the Kalman filter.

As a result, this paper describes

In conclusion, the contribution of this chapter is in providing three new practically implementable filters for stochastic dynamic systems which include nonlinearities in their structure.

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